The Complex Wave Representation (CWR) of Shape

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Over the past three decades, shape analysis has borrowed numerous formalisms, methodologies and techniques from classical physics. Curiously, there has been very little interest in adapting approaches from quantum mechanics. This is despite the fact that *linear* Schrödinger equations are the quantum counterpart to *nonlinear* Hamilton-Jacobi equations [1] and the knowledge that the quantum approaches the classical as Planck's constant $\hbar$ tends to zero.

Distance transforms are popular shape representations. The distance transform scalar field in two dimensions $S(x, y)$ satisfies the static, nonlinear Hamilton-Jacobi equation $\|\nabla S(x, y)\| = 1$. A peculiar fact is that the nonlinear Hamilton-Jacobi equation can be embedded in a linear Schrödinger equation $-\hbar^2 \nabla^2 \psi(x, y) = \psi(x, y)$. The wave function $\psi(x, y)$ is the *complex wave representation* (CWR) of the distance transform. Now, take the Fourier transform of the CWR $\psi(x, y)$ to get $\Psi_\hbar(u, v)$. We see in Figure 1 (the center right figure) that the Fourier transform values lie mainly on a circle and we have observed that this behavior tightens as $\hbar \to 0$. The *stationary phase approximation* [2] can be invoked to explain the reason why the Fourier transform $\Psi_\hbar(u, v)$ takes values mainly on the unit circle.

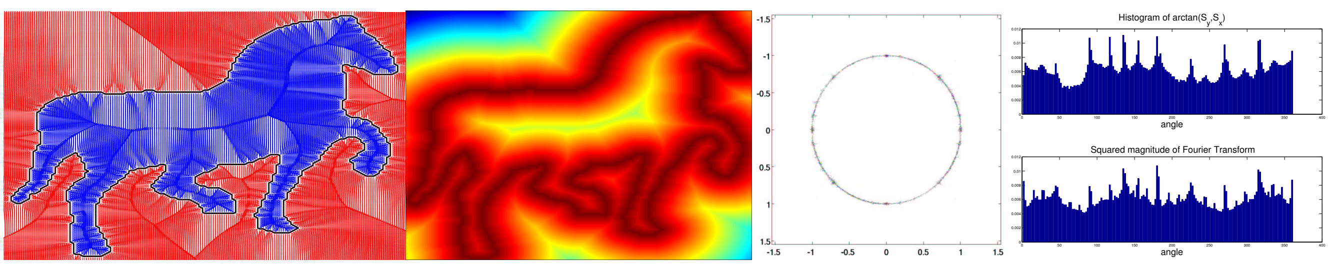

Figure 1: Far left: Distance transform and gradient map of horse silhouette ($\hbar = 0.00004$). Center left: The CWR of the shape. Center right: The scaled and normalized Fourier transform $\Psi_\hbar(u, v)$. Far right: Orientation histogram and the squared magnitude of the Fourier transform. Please ZOOM in (especially the horse silhouette) for greater detail.

We now describe a remarkable empirical discovery buttressed by theoretical analysis [Figure 1 (far right figure).] The squared magnitude of the Fourier transform when polled on the unit circle is approximately equal to the density function of the distance transform gradients with the approximation becoming increasingly exact as $\hbar \to 0$.

Main Result: For Euclidean distance functions satisfying $\|\nabla S(x)\| = 1$ almost everywhere,

$$\boxed{\lim_{\tau \to 0} \frac{1}{\tau} \lim_{\hbar \to 0} \int_{\theta}^{\theta+\tau} |\Psi_\hbar(u(\theta), v(\theta))|^2 \, d\theta = p(\theta)} \tag{1}$$

where $p(\theta)$ is the density function of the unit vector distance transform gradients and $\Psi_\hbar(u(\theta), v(\theta))$ is the Fourier transform of the CWR $\psi(x, y)$ evaluated on the unit circle (in the spatial frequency domain). We see that spatial frequencies are essentially gradient histogram bins.

The unusual connection between the Fourier transform of the CWR $\psi(x, y)$ and the distance transform gradient density opens up a new front for shape analysis. For instance, it would be interesting to examine the shape statistics (mean, covariance etc.) of the CWRs. This represents a fruitful avenue for future research.

References

[1] J. Butterfield. On Hamilton-Jacobi theory as a classical root of quantum theory. In A. Elitzur, S. Dolev, and N. Kolenda, editors, *Quo-Vadis Quantum Mechanics*, chapter 13, pages 239–274. Springer, 2005.

[2] F.W.J. Olver. *Asymptotics and Special Functions*. A.K. Peters/CRC Press, 1997.

Topic: (no category). Preference: (no preference).