Weak Recovery Conditions using Graph Partitioning Bounds

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Abstract

We study a weaker formulation of the nullspace property, requiring it to hold only with high probability, given a Gaussian distribution on the nullspace of the coding matrix A. We show that testing these weak conditions means bounding the optimal value of two classical graph partitioning problems: the k-Dense-Subgraph and MaxCut problems. Both problems admit efficient, tight relaxations and we use semidefinite relaxation techniques to produce tractable bounds and test the performance of our results on several families of coding matrices.

1 Introduction

Given $A \in \mathbf{R}^{q\times n}$ and $e \in \mathbf{R}^q$, we focus on conditions under which the solution to the following minimum cardinality problem

$$\begin{array}{ll} \text{minimize} & \mathbf{Card}(x) \\ \text{subject to} & Ax = Ae \end{array} \tag{1}$$

which is a combinatorial problem in $x \in \mathbf{R}^n$, can be recovered by solving

$$\begin{array}{ll} \text{minimize} & \|x\|_1 \\ \text{subject to} & Ax = Ae \end{array} \tag{2}$$

which is a convex program in $x \in \mathbf{R}^n$. Donoho and Huo (2001) or Cohen et al. (2009) among others show that when $\mathbf{Card}(e) \leq k$ and there is a constant $\alpha_k < 1/2$ such that

$$\|x\|_{k,1} \leq \alpha_k \|x\|_1 \tag{3}$$

for all vectors $x \in \mathbf{R}^n$ with $Ax = 0$, then solving the convex problem (2) will recover the global solution to the combinatorial problem (1).

Here, we seek to enforce a weaker version of condition (3), requiring it to hold only with high probability on the nullspace of A. Let us assume for simplicity that $\mathbf{Rank}(A) = q$, and let $F \in \mathbf{R}^{n\times m}$ with $m = n - q$ be a basis for the nullspace of A. We will require that the condition

$$\|Fy\|_{k,1} \leq \alpha_k \|Fy\|_1 \tag{4}$$

be satisfied with high probability, when $y \sim \mathcal{N}(0, \mathbf{I}_m)$.

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We derive explicit conditions for (4) to hold with high probability and show that our conditions are satisfied by Gaussian matrices for near-optimal signal cardinalities. We then show that testing these weak conditions on arbitrary matrices means bounding the optimal value of two classical graph partitioning problems: the k-Dense-Subgraph and MaxCut problems. Both problems admit efficient, tight relaxations and we use these approximation results to show that our weak recovery conditions can be tested in polynomial time at near optimal signal cardinalities. Finally, we test numerical performance on several families of coding matrices.

Topic: learning algorithms.

Preference: oral.

References

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